Lecture 4 - Curious Phenomena

A Puzzle...

As shown in this video, if you place both fingers underneath a long object (such as a meter stick or hammer), and then tried to slide both fingers simultaneously towards the center, you will find that you cannot. Your fingers will alternate, one at a time, as they move towards the center. Explain this alternation, and describe where the fingers must end up on the object.

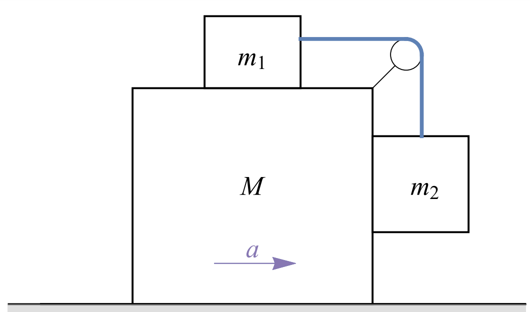
Solution
First, consider where your fingers must end up. When you fingers come together, they hold up the entire weight of the object, and the only way they can prevent the object from toppling over is if they are right under its center of mass (right where gravity applies).

As they travel towards the center, one of your fingers will be closer to the center of mass, and therefore it will have more of the object's weight pressing down upon it, and therefore it will have a larger frictional force. Once it gets close enough to the center of mass, this large frictional force will force this finger to stop moving while the other finger begins to slide along getting closer to the center of mass (recall that kinetic friction is always less than static friction, so once your finger begins to slide it will continue to slide for a while). As stated above, the fingers must ultimately meet at the center of mass. A similar phenomenon occurs with the Rotary Left-Right made by Arvind Gupta. □

Friction Be Gone!

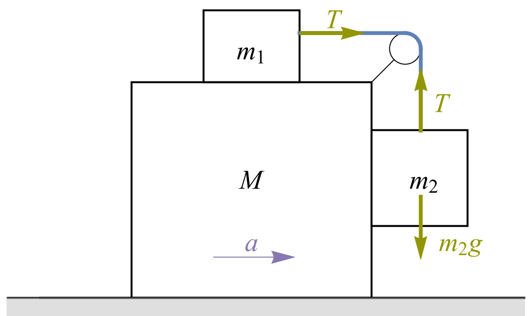
A Strange Equilibrium

Example
Assume all surfaces in the following setup are frictionless, and that the large block M moves with constant acceleration a. For what value of a will the masses m_1 and m_2 not move relative to M?



Solution
The only forces in this problem are gravity, the normal forces, and tension T from the rope. Drawing the relevant forces on the diagram (the gravitational and normal force on m_1 cancel each other while the normal force on m_2

generates m_2's acceleration a to the right),



The top block m_1 must also accelerate horizontally at acceleration a. Then

$$T = m_1 a \tag{1}$$

The second mass must remain in vertical equilibrium, which implies that

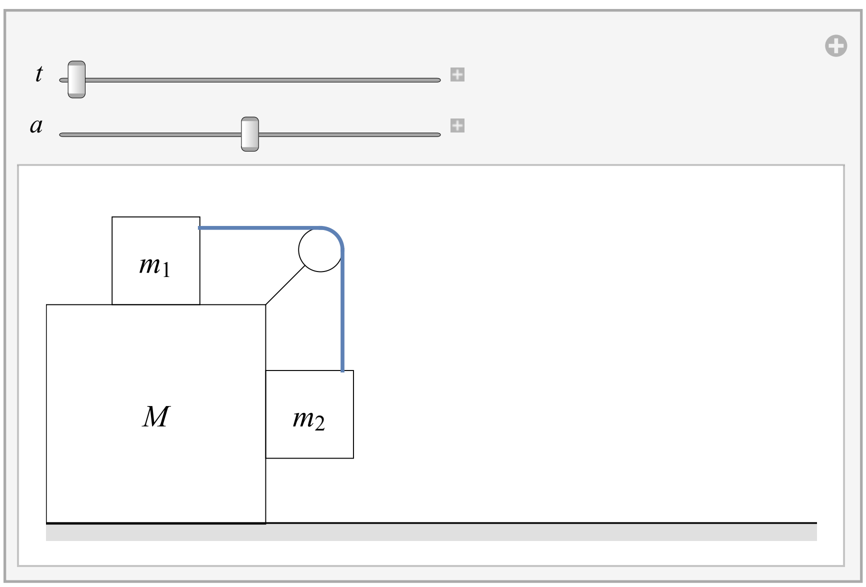
$$T = m_2 g \tag{2}$$

Therefore the acceleration a must satisfy

$$a = \frac{m_2}{m_1} g \tag{3}$$

Note that such an a must exist because for $a = 0$ the mass m_2 would slide down due to gravity, as shown in the Manipulate below. In the other extreme where $a \to \infty$, the mass m_1 would stay relatively stationary as M zooms past it to the right, and because the two masses are connected by a string m_2 will rise up due to this motion. Therefore, for some intermediate a there must be an equilibrium position. □

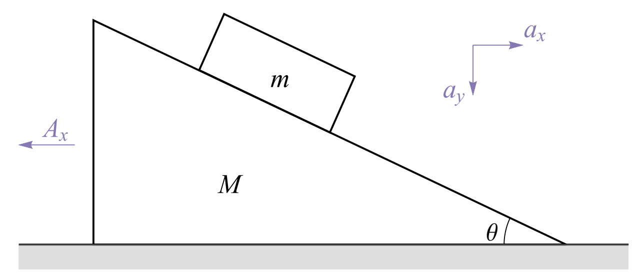
Out[27]=



Sliding Blocks

<u>Example</u>

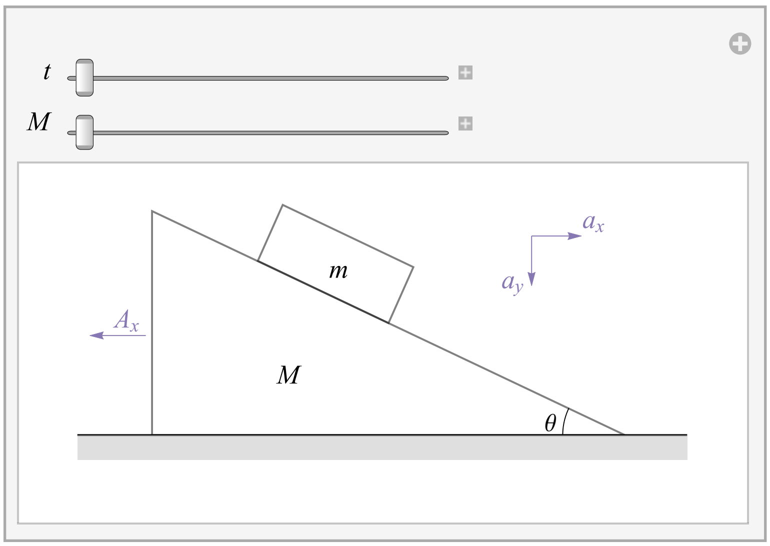
A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane M?



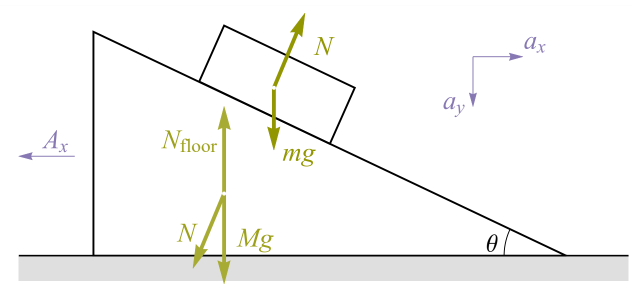
Solution

To build our intuition, consider two limits of the problem that we have already solved before. When $M \to \infty$, the plane will remain stationary while the mass m slides down with an acceleration $g\,\text{Sin}[\theta]$ along its surface. In the opposite limit when $M \to 0$, the mass m will free fall and the mass M will slide out of its way.

Out[28]=



Now let us solve for an arbitrary mass M. Let N be the normal force between the block and the plane. Since friction is not at play in this problem, the only forces on the small block are $m\,g$ and N, while the forces on the plane are $M\,g$, N, and the normal force N_{floor} from the floor.



We denote the horizontal and vertical forces on mass m as a_x and a_y and the horizontal acceleration of the plane M as A_x. Writing $\sum \vec{F} = m\,\vec{a}$ in the $\hat{x}$- and $\hat{y}$-directions,

$$m\,g - N\,\text{Cos}[\theta] = m\,a_y \tag{4}$$

$$N\,\text{Sin}[\theta] = m\,a_x \tag{5}$$

The two $\sum \vec{F} = M \vec{A}$ equations for the large block are

$$N \operatorname{Sin}[\theta] = M A_x \tag{6}$$

$$N \operatorname{Cos}[\theta] + M g - N_{\text{floor}} = 0 \tag{7}$$

Note that this last equation does not give us any useful information. It simply states that the normal force N_{floor} on the plane from the floor has whatever magnitude is necessary to prevent the plane from sinking into the ground.

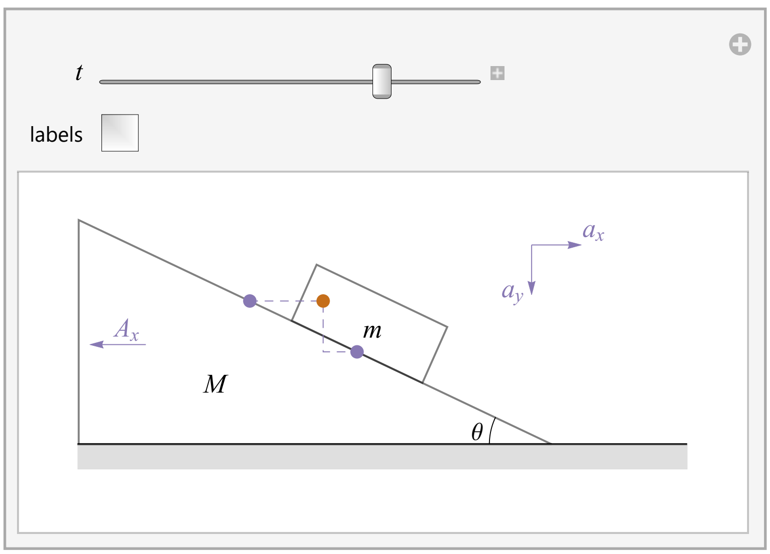
There are 5 unknowns (a_x, a_y, A_x, N, and N_{floor}) so we need another equation. What is it?

The two blocks must stay in contact with each other! This constraint is a bit tricky because the mass m will move downwards and to the right while the plane M will move to the left. Let's focus in on a single point - say the middle point along the bottom of m where it touches the plane M at time 0. This point is shown in brown in the graphics below. Note that this point is *fixed in space* and does not vary with time - it represents where the system started out.

At $t = 0$, the brown point touches a spot (which we label in purple in the graphics below) on the inclined plane. After time t, the inclined plane M will move directly to the left by an amount $\frac{1}{2} A_x t^2$, and so will this spot.

Finally, consider the spot on the mass m which at $t = 0$ touches the brown point. We label this spot in purple as well, and at time t it will have moved vertically down by a distance $\frac{1}{2} a_y t^2$ and to the right by a distance $\frac{1}{2} a_x t^2$.

Out[29]=



In order for the two blocks to remain connected, the two purple points must both be on the inclined plane for all t! In other words,

$$\frac{a_y}{a_x + A_x} = \operatorname{Tan}[\theta] \tag{8}$$

Substituting the other three equations in,

$$\frac{g - \frac{N}{m} \operatorname{Cos}[\theta]}{\frac{N}{m} \operatorname{Sin}[\theta] + \frac{N}{M} \operatorname{Sin}[\theta]} = \operatorname{Tan}[\theta] \tag{9}$$

which we can simplify

$$g - \frac{N}{m} \operatorname{Cos}[\theta] = N \frac{\operatorname{Sin}[\theta]^2}{\operatorname{Cos}[\theta]} \left(\frac{1}{m} + \frac{1}{M}\right) \tag{10}$$

$$g = \frac{N}{\operatorname{Cos}[\theta]} \left(\frac{1}{m} + \frac{\operatorname{Sin}[\theta]^2}{M}\right) \tag{11}$$

$$N = g \operatorname{Cos}[\theta] \left(\frac{1}{m} + \frac{\operatorname{Sin}[\theta]^2}{M}\right)^{-1} \tag{12}$$

Note that in the limit $M \to \infty$, we regain $N = m g \operatorname{Cos}[\theta]$, as expected. We can now substitute this back to find the acceleration of the plane, A_x,

$$\begin{aligned} A_x &= \frac{N \operatorname{Sin}[\theta]}{M} \\ &= \frac{g}{M} \operatorname{Sin}[\theta] \operatorname{Cos}[\theta] \left(\frac{M}{m M} + \frac{m \operatorname{Sin}[\theta]^2}{m M}\right)^{-1} \\ &= \frac{m g \operatorname{Sin}[\theta] \operatorname{Cos}[\theta]}{M + m \operatorname{Sin}[\theta]^2} \end{aligned} \tag{13}$$

Let's check some interesting special cases:

Case 1: $m \gg M$

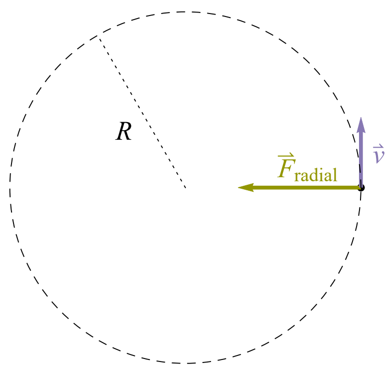
$A_x = \frac{g}{\operatorname{Tan}[\theta]}$ because mass m essentially drops straight down, squeezing the plane M as it goes.

Case 2: $m \ll M$

$A_x = \frac{m g \operatorname{Sin}[\theta] \operatorname{Cos}[\theta]}{M}$ which implies that $a_x = g \operatorname{Sin}[\theta] \operatorname{Cos}[\theta]$. This makes sense because the plane M essentially stands still and the acceleration of m along the plane equals $\frac{a_x}{\operatorname{Cos}[\theta]} = g \operatorname{Sin}[\theta]$, as expected. □

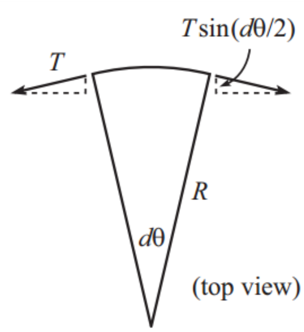
Turning Rope

Recall that a particle travelling at velocity v and acted upon by a radial force $\vec{F}_{\text{radial}}$ (always perpendicular to its velocity) will travel in a circle with radius R satisfying $F_{\text{radial}} = \frac{m v^2}{R}$.



Example

A circular loop of rope with radius R and mass density λ (kg/m) lies on a frictionless table and rotates around its center, with all points moving at speed v. What is the tension in the rope?



Solution

Consider a small piece of rope that subtends an angle $d\theta$. The tension T applies at both ends at a slightly downwards angle of $\frac{d\theta}{2}$ to the horizontal (because the force acts perpendicularly to the radius of the circle). Therefore the radial force felt by this piece of rope equals $2\,T\,\mathrm{Sin}\left[\frac{d\theta}{2}\right] \approx T\,d\theta$ where we have used the small angle approximation $\mathrm{Sin}[x] \approx x$. The mass of this piece of rope equals $m = R\,\lambda\,d\theta$. Equating the centripetal force with $\frac{m\,v^2}{R}$,

$$T\,d\theta = \frac{(R\,\lambda\,d\theta)\,v^2}{R} \tag{14}$$

which simplifies to

$$T = \lambda\,v^2 \tag{15}$$

Note that this independence of R is very interesting! It implies that for any arbitrarily shaped piece of rope, if it moves along its own shape at speed v then its tension equals $\lambda\,v^2$ everywhere! □

Chain Fountain

On February 20, 2013, Steve Mould posted a startling YouTube video demonstrating that a chain being pulled out of a contained. This phenomenon was called the chain fountain, and physicists quickly realized that it defied the simple explanations you might imagine for such a simple system.

An explanation of the chain fountain was can be found in YouTube form or in a paper published in January 2014. The key to the explanation centers on Equation (15)!

Advanced Section: Calculus underlying Circular Motion

Mathematica Initialization